Checking the Consistency of Combined Qualitative Constraint Networks *

Quentin Cohen-Solal and **Maroua Bouzid** and **Alexandre Niveau**
Normandie Univ, UNICAEN, ENSICAEN, CNRS, GREYC, 14000 Caen, France
{quentin.cohen-solal, maroua.bouzid, alexandre.niveau}@unicaen.fr

Abstract

We study the problem of consistency checking for constraint networks over combined qualitative formalisms. We propose a framework which encompasses loose integrations and a form of spatio-temporal reasoning. In particular, we identify sufficient conditions ensuring the polynomiality of consistency checking, and we use them to find tractable subclasses.

1 Introduction

Temporal and spatial reasoning is omnipresent in our daily lives. Computers can achieve them using quantitative approaches; however, for human-computer interaction, quantitative data is often unavailable or unnecessary. This is why research has been carried out about *qualitative* approaches to temporal and spatial reasoning – such as the interval algebra of Allen (1983) – not only in artificial intelligence but also in geographical information systems, databases, and multimedia (Chittaro and Montanari 2000; Chen et al. 2015). Some recent research has focused on the combination of qualitative approaches in order to increase their number of applications. One of the most popular combinations is *loose integration* (Wölfl and Westphal 2009). Spatio-temporal formalisms are other kinds of combinations (Ligozat 2013), allowing the processing of temporal sequences of spatial information (Westphal et al. 2013). *Multi-scale reasoning*, the ability to reason at different levels of detail, is also a form of combination (Hobbs 1985; Li and Nebel 2007; Cohen-Solal, Bouzid, and Niveau 2015).

This paper introduces a formal framework capturing the common structure of loose integrations, multi-scale representations, and temporal sequences of spatial information; all of these can indeed be seen as tuples of *constraint networks* having interdependencies. With loose integrations, each network is based on a different formalism, whereas with multi-scale and spatio-temporal representations all networks are based on the same formalism but hold on different scales and different time periods, respectively. Moreover, in each case, constraints in one network of the tuple can entail constraints between the same variables in the other networks. The entailed constraints correspond to how the initial constraints are transformed by formalism change, scale change, or temporal transition, respectively.

We study in particular *consistency checking* in the context of our framework; we focus on general results that are common to the three kinds of combination, using a simple, well-known instance of a loose integration as a running example. Specifically, we identify sufficient conditions so that the generalized *algebraic closure* can be used to check the consistency of networks over some subclass – which is therefore *tractable*. To sum up, we propose a framework for representing knowledge, reasoning, and identifying tractable fragments, in a unified way, for the three kinds of combination; however, for space reasons, this paper only applies it to loose integrations and spatio-temporal sequences.

We begin by recalling concepts related to temporal and spatial formalisms, then we give some background on combinations of formalisms, notably loose integrations and spatio-temporal sequences. Section 3 introduces our framework and Section 4 establishes our tractability results, which are then illustrated on the combination of size and topology.

2 Background and Related Work

Qualitative Temporal and Spatial Formalisms

In the context of qualitative temporal and spatial reasoning, we are particularly interested in checking the consistency of temporal or spatial *descriptions*, encoded by relations between spatial or temporal *entities* of a set U. Each relation is a set of *basic* relations from a set $\mathcal{B}$: this represents the uncertainty about the actual basic relation – e.g., $x\ \{<,=\}\ y$ means that either $x < y$ or $x = y$. The set of all relations forms a *non-associative relation algebra* $\mathcal{A} = 2^{\mathcal{B}}$ (Ligozat 2013, Ch. 11). Well-known algebras include the interval algebra of Allen (1983), but also the *point algebra* PA (Vilain, Kautz, and van Beek 1989), whose basic relations are $\mathcal{B}_{\text{PA}} = \{<,=,>\}$, and the algebra RCC8 of *topological relations* (Randell, Cui, and Cohn 1992), whose basic relations are described in Fig. 1. There are several operators over relations in $\mathcal{A}$: *inverse* of a relation r, denoted by $\bar{r}$, *intersection* of r_1 and r_2, denoted by $r_1 \cap r_2$, and *(weak) composition* of r_1 and r_2 (Renz and Ligozat 2005), denoted by $r_1 \diamond r_2$. These operators allow one to reason, by reducing the uncertainty about basic relations of entities: $x\ r\ y \iff y\ \bar{r}\ x$;

*This is a revised version of the article published at AAAI 2017 (Definition 9 and the proof of Corollary 27 have been fixed).

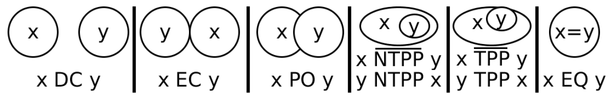

Figure 1: The 8 relations of RCC8 in the plane.

$x\ r_1\ y \wedge x\ r_2\ y \iff x\ (r_1 \cap r_2)\ y$; and $x\ r_1\ y \wedge y\ r_2\ z \implies x\ (r_1 \diamond r_2)\ z$. For example, if we know that $x\{\mathrm{EC}\}y$ and $y\{\mathrm{NTPP}\}z$, we can deduce that $x\{\mathrm{PO}, \mathrm{NTPP}, \mathrm{TPP}\}z$.

A temporal or spatial description can be modeled by a *qualitative constraint network* – a labeled graph in which nodes are *entity variables* and edges are labeled with a binary relation of $\mathcal{A}$. More formally, a qualitative constraint network N over a set of relations $\mathcal{S} \subseteq 2^{\mathcal{B}}$ is a pair $N = (\mathtt{E}, \mathtt{C})$ where $\mathtt{E}$ is a finite set of entity variables of U and $\mathtt{C}$ is a set of *constraints over* $\mathcal{S}$, i.e., tuples (x, r, y) with $x, y \in \mathtt{E}$, $x \neq y$ and $r \in \mathcal{S}$. An example of a network over RCC8 is $N = (\{x, y, z\}, \{(x, \{\mathrm{EC}\}, y), (y, \{\mathrm{NTPP}\}, z)\})$. Like all networks in this paper, it is normalized, in the sense that for each pair $\{x, y\}$ there is at most one constraint, whose relation is denoted N^{xy} (implicitly, $N^{yx} = \overline{N^{xy}}$). If there is no constraint between two entities, the constraint is implicit, and the corresponding relation is the whole set $\mathcal{B}$. We say that a network N *refines* another network N' if it holds that $\forall x, y \in \mathtt{E}\colon N^{xy} \subseteq (N')^{xy}$, which we denote by $N \subseteq N'$.

The notion of *solution* of a constraint network depends on a semantics, which is given by an *interpretation function* φ mapping any relation r of the algebra to the set of all pairs of entities from the domain U *satisfying* r. When φ verifies specific properties, the triple $(\mathcal{A}, U, \varphi)$ constitutes a *qualitative formalism* (Ligozat 2013, Ch. 11). Thus, a solution of a constraint network N is a set $\{u_x\}_{x \in \mathtt{E}} \subseteq U$ such that $\forall x, y \in \mathtt{E}\colon (u_x, u_y) \in \varphi(N^{xy})$. A fundamental problem is to determine whether a constraint network has at least one solution, in which case it is said to be *consistent*. To each solution of a network N corresponds a unique *scenario* of N, i.e., a network $S \subseteq N$ such that $\forall x, y \in \mathtt{E}\colon S^{xy} \in \mathcal{B}$. For instance, our example RCC8 network is not a scenario (N^{xz} is not basic). Finding a solution of a network N amounts to finding a consistent scenario S of N, since any solution of S is a solution of N. Because consistency checking is NP-complete for many algebras, some research focuses on *tractable subclasses*, i.e., sets $\mathcal{S} \subseteq 2^{\mathcal{B}}$ that are closed under intersection, weak composition and inversion, such that it is polynomial to decide the consistency of any network whose relations are in $\mathcal{S}$ (see Ligozat 2013).

A constraint network is *algebraically closed* – a key concept to find consistent scenarios in a purely algebraic way – if $N^{xz} \subseteq N^{xy} \diamond N^{yz}$ for all $x, y, z \in \mathtt{E}$. We can obtain from any network N an algebraically closed network having the same solutions by computing its *algebraic closure*. It can be done (in polynomial time) by repeatedly replacing each N^{xz} by $(N^{xy} \diamond N^{yz}) \cap N^{xz}$ until a fixed point is reached. If the resulting network is not trivially inconsistent (i.e., if none of its relations is the empty set), it is said to be $\diamond$*-consistent*. In the literature, $\diamond$-consistency is often conflated with *path-consistency*, because for some formalisms they are equivalent (Renz and Ligozat 2005). An algebraically closed *scenario* is always $\diamond$-consistent by definition, but note that it is not necessarily consistent for any formalism (Renz and Ligozat 2005). However, when all the algebraically closed scenarios of a formalism are consistent, the consistency of a network can be decided by searching for an algebraically closed scenario, using backtracking methods based on algebraic closure (Ladkin and Reinefeld 1992). For some subclasses, any $\diamond$-consistent network is consistent, so there is no need to backtrack: such subclasses are thus tractable.

Combined Spatial and Temporal Formalisms

Some research has recently been focusing on combining qualitative formalisms. One of these combinations, which our framework encompasses, is the *loose integration* of two qualitative formalisms and its *biconstraint networks* (two networks having interdependencies) (Westphal and Woelfl 2008). The consistency checking problem is then to decide whether there is a solution satisfying both networks.

Example 1. The loose integration of qualitative size and topology of Gerevini and Renz (2002), which we call QST, describes the relation between two regions both in terms of topology and in terms of their relative size; e.g., "*x* and *y* are disjoint and the size of *x* is smaller than that of *y*".

To reason on QST, Gerevini and Renz generalized the path-consistency algorithm, which simply computes the algebraic closure, into the *bipath-consistency* algorithm, which enforces $\diamond$-consistency on both networks while simultaneously propagating their interdependencies. Subclasses for which bipath-consistency decides consistency have been found for several combinations of formalisms (Gerevini and Renz 2002; Li and Cohn 2012; Cohn et al. 2014).

The framework introduced in this paper encompasses loose integrations (generalized to *m* formalisms) as specific combinations, but does not cover all ways of combining formalisms. *Tight integrations* (Wölfl and Westphal 2009) are more expressive than loose integrations, at the cost of drastically increasing the number of relations. Another combination is that of Meiri (1996), which deals with *heterogeneous* entities that are points and intervals. The corresponding relations are relations between a point and an interval, two intervals, and two points, respectively. In this combination, whose complexity has been studied in depth by Jonsson and Krokhin (2004), there is only one relation per pair of entities; in constrast, loose integrations feature several relations (from different formalisms) between the same entities, which increases expressiveness by allowing the use of complementary relations. This complementarity is the main asset of loose integration (and its major difficulty). Note that there also exist combinations with non-qualitative formalisms (Meiri 1996; Bennett et al. 2002)

Spatio-Temporal Formalisms

Spatio-temporal formalisms are also combinations, which integrate space and time information in particular ways.

Westphal et al. (2013) proposed a method to reason about temporal sequences of spatial information, which actually share the same structure as loose integrations and are thus covered by our framework. They model such sequences as

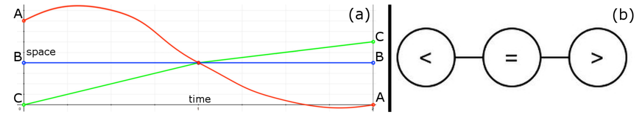

Figure 2: (a) An evolution of space points. (b) The neighborhood graph of the point algebra.

tuples of constraint networks, each corresponding to a time instant. They introduce two kinds of solutions, depending on the desired dynamics of entities (moving continuously) over time. Our framework covers the weaker "T_2-solutions", which guarantee that between successive instants of the sequence, for each pair of entities, only the relation of the first instant and then the relation of the second instant hold.

Example 2. The following is a temporal sequence of 3 networks describing spatial points moving along a line: $x \leq y < z$ at the first instant, $x = y = z$ at the second, and then $x > y > z$ at the third instant. This description has temporally continuous solutions without intermediary relations between the instants, such as that of Figure 2 (a).

In fact, the T_2 condition forces relations at successive instants to be "neighbors" according to the *neighborhood graph* of PA (Freksa 1991), shown in Figure 2 (b). In this graph, for example, the only neighbor relation of "$<$" is "$=$". The T_2 condition thus ensures that, if $x < y$ at one instant, then at any neighbor instant, either $x < y$ or $x = y$.

Gerevini and Nebel (2002) proposed a similar formalism based on time intervals but with uncertainty on the scheduling of intervals, so it is not encompassed by our framework.

3 Representation and Reasoning with Multi-Algebras

This section introduces *multi-algebras*, which constitute the underlying structure of loose integrations and temporal sequences. It shows how one can reason about multi-algebra relations, then provides them with a formal semantics, and finally generalizes constraint networks and their algebraic closure to this broader setting.

Projections and Multi-Algebras

Let us introduce the general building blocks of our framework, beginning with *projections*, which aim at representing the interdependencies of relations from different formalisms. The projection of a relation r from a formalism onto another is the set of basic relations of the other formalism which may hold given that r holds.

Definition 3. Let $\mathcal{A} = 2^{\mathcal{B}}$ and $\mathcal{A}' = 2^{\mathcal{B}'}$ be two algebras. A *projection operator* is a function $\Rsh : \mathcal{A} \to \mathcal{A}'$ which satisfies (i) $\forall b \in \mathcal{B},\ \Rsh\{\bar{b}\} = \overline{\Rsh\{b\}}$, and (ii) $\forall r \in \mathcal{A},\ \Rsh r = \bigcup_{b \in r} \Rsh\{b\}$.

We can now define *multi-algebras*, the key objects of our framework, which are Cartesian products of algebras (each corresponding to one of the combined formalisms, or one instant in a temporal sequence) associated with projection operators representing the interdependencies of their relations.

Definition 4. A *multi-algebra* $\mathcal{A}$ is the Cartesian product of m algebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$ (with $m \in \mathbb{N}^*$), equipped with $m(m-1)$ projection operators $\Rsh_i^j : \mathcal{A}_i \to \mathcal{A}_j$ (for any distinct $i, j \in \{1, \ldots, m\}$). We call *relations* the elements R of $\mathcal{A}$, although they are actually m-tuples of relations; R_i denotes the (classical) relation associated with $\mathcal{A}_i$ in R. We say that R is *basic* when all R_i are basic ($R \in \mathcal{B}_1 \times \cdots \times \mathcal{B}_m$).

Note that a multi-algebra with $m = 1$ ("mono-algebra") is exactly a classical algebra, as it has no projection operators.

Example 5. The multi-algebra corresponding to QST (see Ex. 1) is the Cartesian product RCC8 $\times$ PA of the RCC8 algebra (see Fig. 1) and the point algebra PA (for region sizes), with the interdependency operators of QST as projections. One of its relations is $(\{\mathrm{TPP}\}, \{<, =\})$, and the projection of $\{\mathrm{TPP}\}$ into PA is $\Rsh_{\mathrm{RCC8}}^{\mathrm{PA}}\{\mathrm{TPP}\} = \{<\}$ (since TPP is the "tangential proper part" relation and a region strictly included in another always has a smaller size).

Example 6. The multi-algebra PA^m can be used to represent sequences of binary relations between points on a line, as in Ex. 2: the i^{th} PA corresponds to the i^{th} instant of the sequence, thus R_i is the relation at instant i. The projections enforcing the neighborhood graph of Fig. 2 are $\Rsh_i^j\{<\} = \{<, =\}$, $\Rsh_i^j\{>\} = \{>, =\}$, and $\Rsh_i^j\{=\} = \mathcal{B}$, if instants i and j are neighbors (i.e., $|i - j| = 1$), and $\forall b \in \mathcal{B}\colon\ \Rsh_i^j\{b\} = \mathcal{B}$ (i.e., no constraint), if they are not. For instance, PA $\times$ PA $\times$ PA is the multi-algebra corresponding to three instants. The relation $(\{<, =\}, \{=\}, \{>\})$ of this multi-algebra represents a possible 3-instant sequence of relations.

When there is no ambiguity, we use a lighter notation for relations, thus writing $(\{\mathrm{TPP}, \mathrm{EQ}\}, \{<\})$ as $(\mathrm{TPP\ EQ}, <)$, and $(\{<, =\}, \{=\}, \{<, >\})$ as $(\leq, =, \neq)$, for example.

Reasoning about Multi-Algebra Relations

We can reason about multi-algebra relations by applying the classical rules componentwise: for instance, in QST (Ex. 5), if $x\ (\mathrm{TPP}, \leq)\ y$ and $y\ (\mathrm{DC}, =)\ z$, then $x\ (\mathrm{DC}, \leq)\ z$ (since TPP $\diamond$ DC is DC and $\leq \diamond =$ is $\leq$). It is thus natural to introduce composition $\diamond$, intersection $\cap$, and inversion $\bar{\cdot}$ operators over multi-algebra relations which simply work componentwise (e.g., $(R \diamond R')_i = R_i \diamond R'_i$). These operators are also useful to apply classical concepts to our generalized framework (for the same reason, we also write $R \subseteq R'$ if $R_i \subseteq R'_i$ for each i). They are, however, not sufficient for reasoning: we also need to propagate the interdependencies inside each relation.

Definition 7. The *projection closure* of $R \in \mathcal{A}$, denoted by $\Rsh(R)$, is obtained from R by repeatedly replacing each R_j by $R_j \cap (\Rsh_i^j R_i)$ for all distinct i, j until a fixed point is reached.

Example 8. In QST, since $\Rsh_{\mathrm{RCC8}}^{\mathrm{PA}}\{\mathrm{TPP}\} = \{<\}$ (Ex. 5), the projection closure of $(\mathrm{TPP}, \leq)$ is $\Rsh(\mathrm{TPP}, \leq) = (\mathrm{TPP}, <)$, and also $\Rsh(\mathrm{TPP}, \geq) = (\varnothing, \varnothing)$, which proves that this relation is not feasible (indeed, a region cannot be inside another while having a larger surface). In the context of Ex. 6, $\Rsh(<, \neq, >) = (\varnothing, \varnothing, \varnothing)$, so this 3-instant sequence is not feasible.

Semantics and Consistency of Relations

Using these operators, we can now give multi-algebra relations a proper semantics, taking an approach similar to the classical case. For this, we set $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_m)$ and $B \in R \iff B \subseteq R \wedge B \in \mathcal{B}_1 \times \cdots \times \mathcal{B}_m$, i.e. B is a basic relation of R.

Definition 9. A *(loosely) combined qualitative formalism* is a triple $(\mathcal{A}, U, \varphi)$, where $\mathcal{A}$ is a multi-algebra, U is an entity domain, and the interpretation $\varphi \colon \mathcal{A} \to 2^{U \times U}$ satisfies:[1]

$$\varphi(\Rsh R) = \varphi(R) \qquad \varphi(R \diamond R') \supseteq \big(\varphi(R) \circ \varphi(R')\big) \cap \varphi(\mathcal{B})$$

$$\varphi(\bar{R}) = \overline{\varphi(R)} \qquad \varphi(R \cap R') = \varphi(R) \cap \varphi(R')$$

$$\varphi\big((\varnothing, \ldots, \varnothing)\big) = \varnothing \qquad \varphi(R) = \bigcup_{B \in R} \varphi(B)$$

with $R, R' \in \mathcal{A}$ and $\circ$ the true composition of relations on U.

These straightforward requirements ensure that the operators are *sound* (i.e., do not remove valid pairs of entities), and that a relation is *consistent* (i.e., has a nonempty interpretation) if and only if it contains a consistent basic relation.

Example 10. We call *temporalized point calculus (*TPC*)* the combined formalism representing temporal sequences of the point algebra interpreted spatially. Its multi-algebra is simply PA^m (for sequences of length m), as described in Ex. 6. Its interpretation function (which we cannot define formally for space reasons) associates with each relation the set of pairs of points evolving continuously on $\mathbb{R}$ along the time of the sequence, satisfying at each instant of the sequence the corresponding relation, and not satisfying other relations between these instants (this is the T_2 condition; see Ex. 2).

The combined formalism corresponding to the loose integration of m formalisms is straightforward to define (it can be checked that all requirements of Def. 9 are satisfied):

Definition 11. The *loose integration* of m qualitative formalisms $(\mathcal{A}_1, U, \varphi_1), \ldots, (\mathcal{A}_m, U, \varphi_m)$ over the same domain U is the combined formalism $(\mathcal{A}, U, \varphi)$, where $\mathcal{A}$ is the multi-algebra $\mathcal{A}_1 \times \cdots \times \mathcal{A}_m$ with each $\Rsh_i^j$ satisfying $\forall b \in \mathcal{B}_i \colon \Rsh_i^j\{b\} = \{b' \in \mathcal{B}_j \mid \varphi_i(b) \cap \varphi_j(b') \neq \varnothing\}$, and $\varphi \colon (r_1, \ldots, r_m) \mapsto \varphi_1(r_1) \cap \cdots \cap \varphi_m(r_m)$.

Example 12. QST is exactly the loose integration of the RCC8 formalism and of the formalism interpreting PA in terms of region sizes; we recover the multi-algebra of Ex. 5. On the other hand, it can be shown that TPC (Ex. 10) is not a loose integration (the interpretations at every instant cannot be defined independently from one another).

Now, while in classical formalisms any basic relation is consistent, this is no longer the case in combined formalisms: because of interdependencies, multi-algebra relations (even basic ones) can be *inconsistent*. In particular, we have seen (Ex. 8) that the projection closure of a relation can

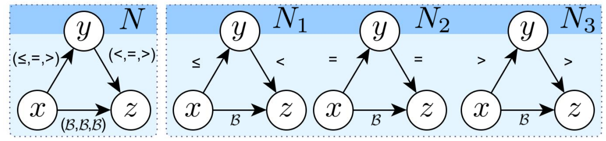

Figure 3: A network N over PA^3 and its three slices.

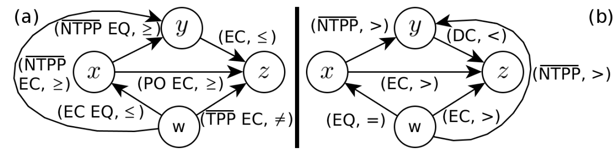

Figure 4: (a) An algebraically consistent, yet inconsistent, QST network. (b) A consistent scenario (see Ex. 16).

be empty. Closing a relation under its projection operators can thus help detecting its inconsistency: if $\Rsh R$ is empty, we can conclude that R is inconsistent. Otherwise, $\Rsh R$ is "consistent with respect to projections", or $\Rsh$*-consistent*.

Definition 13. A multi-algebra relation R is $\Rsh$*-consistent* if $R = \Rsh(R)$ and $R_i \neq \varnothing$ for each i.

However, observe that while projections can remove *pairwise* inconsistencies, consistency ultimately depends on the interpretation function. Hence, $\Rsh$-consistency does not imply consistency in general, although it is the case for some combined formalisms: we can prove in particular that any $\Rsh$-consistent relation of TPC and QST (Ex. 10, 12) is consistent. In such cases, closing a relation by projection suffices to check its consistency.

Multi-Algebra Networks and Algebraic Closure

We simply model descriptions from combined formalisms as *qualitative constraint networks over multi-algebras*, which work exactly like classical networks except that constraints between entity variables are m-tuples of relations. For example, the network N in Fig. 3 corresponds to the temporal sequence of 3 networks over the point algebra shown in Ex. 2; relation N^{xy} is the sequence of relations between x and y. It is important to note that N can equivalently be seen as a tuple of classical networks over each $\mathcal{A}_i$, as shown in Fig. 3, where N_i is the network at instant i:

Definition 14. Let $N = (\mathtt{E}, \mathtt{C})$ be a network over some multi-algebra $\mathcal{A}$. The i^{th} *slice of* N, denoted by N_i, is the network $(\mathtt{E}, \mathtt{C}_i)$ over $\mathcal{A}_i$, where $\mathtt{C}_i = \{(x, R_i, y) \mid (x, R, y) \in \mathtt{C}\}$.

We directly adapt the notions of *solution*, *consistency* and *scenario* (Sect. 2) to networks over multi-algebras (with respect to a combined formalism; we often omit this precision when doing so is harmless). For instance, a solution of the network in Fig. 3 is the evolution in Fig. 2 (a). Other notions must be generalized:

Definition 15. A network N over a multi-algebra $\mathcal{A}_1 \times \cdots \times \mathcal{A}_m$ is *trivially inconsistent* if $\exists i \in \{1, \ldots, m\} \colon \exists x, y \in \mathtt{E} \colon N_i^{xy} = \varnothing$. It is *algebraically closed* if $N^{xz} \subseteq N^{xy} \diamond N^{yz}$ and $N^{xy} = \Rsh N^{xy}$ for all $x, y, z \in \mathtt{E}$. It is *algebraically consistent* if it is algebraically closed and not trivially inconsistent.

[1]Mistaken axiom $\varphi(R \cup R') = \varphi(R) \cup \varphi(R')$ has been fixed in this revised version. This property of qualitative formalisms is indeed lost by loose integration: for example, relations of QST $(\{\mathrm{DC}\}, \varnothing)$ and $(\varnothing, \{=\})$ have an empty interpretation, although the interpretation of $(\{\mathrm{DC}\}, \{=\})$ is nonempty.

For networks over multi-algebras, being algebraically closed is being closed under both composition *and* projection. This cleanly generalizes the classical case, since any relation of a mono-algebra is vacuously closed under projection.

Algebraic consistency generalizes bipath-consistency to m dimensions, and works in the same way – it is a necessary condition for consistency that can be used to filter out inconsistent networks. It can be enforced by alternately closing each relation N^{xy} under projection and each slice N_i under composition until a fixed point is reached.

Example 16. The network N over QST in Fig. 4 (a) is algebraically consistent. However, if we remove "$=$" from N_{PA}^{xy}, the network is no longer algebraically closed because it is not closed under projection, as $\{\overline{\text{NTPP}}, \text{EQ}\} \nsubseteq \Rsh_{\text{PA}}^{\text{RCC8}}\{>\}$ (since clearly $\text{EQ} \notin \Rsh_{\text{PA}}^{\text{RCC8}}\{>\}$). Now, while N is algebraically consistent (and although N_{PA} and N_{RCC8} are consistent), it is actually inconsistent: First, "$=$" of N_{PA}^{xy} does not belong to any consistent scenario of N_{PA} (van Beek and Cohen 1989, p. 13). Second, $\overline{\text{NTPP}}$ of N^{xy} is not feasible either (for N_{RCC8}). The only remaining relation, $x\ (\text{EQ}, >)\ y$, is also not feasible, since its projection closure is empty.

However, by adding DC to N_{RCC8}^{yz}, the network remains algebraically consistent, but this time it becomes consistent. Indeed, Fig. 4 (b) shows one of its consistent scenarios.

4 Tractability Results

Let us now study the problem of checking the consistency of networks over multi-algebras. Since this problem is NP-complete for many formalisms, we proceed as in the classical case: we focus on subsets, and notably on *subclasses*, of multi-algebras. Then, we present two theorems providing conditions under which a subclass is *tractable*.

Algebraically Tractable Subclasses

We first introduce two kinds of multi-algebra subsets, namely *subclasses* and the more specific *subalgebras*:

Definition 17. A *subclass* of a multi-algebra $\mathcal{A}$ is a set of relations $\mathcal{S} \subseteq \mathcal{A}$ which is closed under componentwise composition, intersection, and inversion. If $\mathcal{S}$ contains the basic relations (i.e., $\mathcal{B}_1 \times \cdots \times \mathcal{B}_m \subseteq \mathcal{S}$), we call it a *subalgebra*.

For example, $\text{H}_8 \times \text{PA}$ – where H_8 is a well-known subalgebra of RCC8 (Gerevini and Renz 2002) – is a subalgebra of the multi-algebra $\text{RCC8} \times \text{PA}$. Subalgebras are particularly interesting subclasses since all scenarios of the multi-algebra are scenarios of these subclasses. Moreover, the most studied subclasses are subalgebras (Nebel and Bürckert 1995; Ligozat 1996; Renz 1999; Long and Li 2015).

The following notion of *slice* of a multi-algebra subset is a kind of reverse operator of the Cartesian product. It will allow us to lift tractability results from the classical setting to the multi-algebra setting.

Definition 18. The i^{th} *slice* of a multi-algebra subset $\mathcal{S} \subseteq \mathcal{A}$, denoted $\mathcal{S}_i$, is the subset of $\mathcal{A}_i$ defined by $\mathcal{S}_i = \{R_i \mid R \in \mathcal{S}\}$.

Note that $\mathcal{S}$ is a subset of $\mathcal{S}_1 \times \cdots \times \mathcal{S}_m$, the Cartesian product of its slices. It is also not hard to see that, when $\mathcal{S}$ is a subclass, each slice $\mathcal{S}_i$ is a subclass of $\mathcal{A}_i$, and the Cartesian product of the $\mathcal{S}_i$ is also a subclass of $\mathcal{A}$.

Recall that the algebraic closure is classically used to detect inconsistent networks, providing a consistency checking procedure that is polynomial and sound (since the operators are sound, thanks to Def. 9), but incomplete. We focus on subclasses for which the procedure is complete:

Definition 19. A subclass $\mathcal{S}$ is said to be *algebraically tractable* when, for any network N over $\mathcal{S}$, if the algebraic closure of N is not trivially inconsistent then N is consistent.

Clearly, for a subclass to be algebraically tractable, all the algebraically closed scenarios over this subclass must be consistent; this depends on the interpretation function of the combined formalism. For instance, some algebraically closed scenarios over the combination of RCC8 (with weak connectedness) and the rectangle algebra are inconsistent (Cohn et al. 2014); consequently, no *subalgebra* of this combination can be algebraically tractable.

One could think that if all algebraically consistent networks over a subclass $\mathcal{S}$ are consistent, then $\mathcal{S}$ is algebraically tractable; but this is not sufficient because, contrary to the classical case, the algebraic closure of a network over a subclass $\mathcal{S}$ is not necessarily over $\mathcal{S}$. It clearly becomes sufficient if $\mathcal{S}$ is $\Rsh$*-closed*, i.e., if the projection closure of any relation of $\mathcal{S}$ is in $\mathcal{S}$ ($\forall R \in \mathcal{S}\colon \Rsh R \in \mathcal{S}$).

Proposition 20. *A $\Rsh$-closed subclass over which algebraically consistent networks are consistent is algebraically tractable.*

Now, under which conditions does a $\Rsh$-closed subclass verify that its algebraically consistent networks are consistent? In the following we state two complementary theorems providing such conditions. With the first theorem, a subalgebra inherits its tractability from that of its slices, whereas with the second theorem, tractability is inherited from a smaller subset of relations by *refinement*.

Inheriting Tractability from Subalgebra Slices

In this section, we focus on conditions ensuring that a subalgebra $\mathcal{S}$ is tractable by using the tractability of its slices $\mathcal{S}_i$. One of the conditions is that each slice be *scenarizable by a refinement* (a *refinement* of a multi-algebra subset $\mathcal{S}$ is a mapping $h\colon \mathcal{S} \to \mathcal{A}$ such that $h(R) \subseteq R$ for all $R \in \mathcal{S}$).

Definition 21. A mono-subalgebra $\mathcal{S}$ is *scenarizable by a refinement h* if for any $\diamond$-consistent network N over $\mathcal{S}$ and any $x, y \in \mathtt{E}$, (i) $h(N^{xy}) \neq \varnothing$ and (ii) for any $b \in h(N^{xy})$ there exists an algebraically closed scenario $S \subseteq N$ such that $S^{xy} = b$.

When all the algebraically closed scenarios over $\mathcal{S}$ are consistent (see remark after Def. 19), this property actually entails the algebraic tractability of $\mathcal{S}$. Indeed, from any $\diamond$-consistent network over $\mathcal{S}$, we can obtain a consistent scenario by (i) choosing a pair of variables, (ii) replacing their relation by a basic relation of the refinement, (iii) computing the algebraic closure – and repeating these steps until a scenario is obtained. Finding a refinement by which a subalgebra $\mathcal{S}$ is scenarizable is a classical method to prove that $\mathcal{S}$

is tractable. For example, the point algebra PA and the pre-convex subclass of the interval algebra are scenarizable by $h_{\max}$, the refinement by the basic relations of "maximal dimension" (Ligozat 2013, Ch. 2): $h_{\max}(r) = \{b \in r \mid \dim(b) = \dim(r)\}$.

Now, considering a subalgebra $\mathcal{S}$ whose each slice $\mathcal{S}_i$ is scenarizable by some h_i, a natural idea would be to apply the classical technique by combining $h_1, \ldots, h_m$ into a specific form of refinement over multi-algebra relations:

Definition 22. A *multi-refinement* of a multi-algebra subset $\mathcal{S}$ is a refinement of the form $H = (h_1, \ldots, h_m)$ with each h_i a refinement of $\mathcal{S}_i$, defined as $H\colon R \mapsto (h_1(R_1), \ldots, h_m(R_m))$.

However, even if each $\mathcal{S}_i$ is scenarizable by h_i, there is in fact no guarantee that the multi-refinement $H = (h_1, \ldots, h_m)$ can be used to find a consistent scenario using an adaptation of the "scenarizability by h" method. Additional requirements are needed to ensure that the individual refinements work well together with respect to projections. First, the refinement of $\Rsh$-consistent relations (Def. 13) by H must be consistent; but this is not sufficient for any algebraically consistent network to remain consistent after refinement. Consequently, we assume in addition the following property:

Definition 23. A network is *simple* when closing it under projection and then under composition makes it either algebraically consistent or trivially inconsistent.

A subalgebra $\mathcal{S}$ is *simple* if any network over $\mathcal{S}$ is simple.

Since being closed is a local property, it can be shown that enumerating all 3-variable bi-networks over each $\mathcal{S}_i \times \mathcal{S}_j$ suffices to check that a subalgebra $\mathcal{S}$ is simple.

Using the previous properties, we state our first theorem:

Theorem 24 (Slicing theorem)**.** *Let $\mathcal{S}$ be a subalgebra whose algebraically closed scenarios are consistent, and $H = (h_1, \ldots, h_m)$ be a multi-refinement. If we have:*

(C1) each slice $\mathcal{S}_i$ is scenarizable by h_i;

(C2) $\mathcal{S}$ is simple; and

(C3) for any $\Rsh$-consistent R of $\mathcal{S}$, $H(R)$ is consistent;

then algebraically consistent networks over $\mathcal{S}$ are consistent. If, in addition, $\mathcal{S}$ is $\Rsh$-closed, then $\mathcal{S}$ is algebraically tractable.

Proof. Let N be an algebraically consistent network over $\mathcal{S}$, and x, y be variables such that N^{xy} is not basic. We know that $H(N^{xy})$ is consistent (C3), that is, it contains at least one consistent basic relation (Def. 9). We refine N^{xy} by one such relation $B = (b_1, \ldots, b_m)$. Obviously, the modified N, which we denote by N', is still closed under projection.

Moreover, since each slice N_i is $\diamond$-consistent, there exists an algebraically closed scenario $S_i \subseteq N_i$ such that $S_i^{xy} = b_i$ (C1). Therefore, $S_i \subseteq N_i'$ holds for all i, which ensures that the closure of N' under composition is not trivially inconsistent. The closure of N' is thus algebraically consistent, since $\mathcal{S}$ is simple (C2) and N' is closed under projection.

All in all, the result is an algebraically consistent network over $\mathcal{S}$ (as $\mathcal{S}$ is a subalgebra). We can thus apply the procedure from the start once again, iteratively making all relations basic and consistent. The process necessarily ends with an algebraically closed scenario, which is consistent by hypothesis. Hence N is consistent (Def. 9). The second conclusion is a direct corollary (by Prop. 20). □

This theorem can be used to prove that a $\Rsh$-closed subclass $\mathcal{S}$ built from known tractable subalgebras is tractable. It also gives a very efficient way to check the consistency of a network over $\mathcal{S}$, only requiring one projection closure followed by one composition closure. Moreover, the proof describes an efficient algorithm *exhibiting* a consistent scenario.

Inheriting Tractability from a Subset of Relations

Now, we focus on an alternative set of conditions ensuring the tractability of a $\Rsh$-closed subclass $\mathcal{S}$. This result is inspired by the classical technique of "reduction by a refinement" (Renz 1999). The idea is to inherit, from a smaller multi-algebra subset $\mathcal{S}'$, the fact that algebraically consistent networks are consistent – and to conclude with Prop. 20. Indeed, if we can refine networks over $\mathcal{S}$ to networks over $\mathcal{S}'$ while preserving algebraic consistency, then all algebraically consistent networks over $\mathcal{S}$ are consistent. We call this requirement *algebraic stability by a refinement*:

Definition 25. A multi-algebra subset $\mathcal{S}$ is *algebraically stable by a refinement H* if, for any algebraically consistent network N over $\mathcal{S}$, the refined network $H(N)$ (obtained from N by simultaneously replacing each relation N^{xy} by $H(N^{xy})$) is still algebraically consistent.

For example, $\mathrm{H}_8 \times \mathrm{PA}$ is algebraically stable by $H = (h_{\mathrm{H}_8}, h_{\max})$, with h_{H_8} the H_8 refinement of Renz (1999, Lemma 20); like simplicity, stability is easy to check by enumeration.

Our theorem formalizes the reduction mechanism:

Theorem 26 (Refinement theorem)**.** *Let H be a refinement from a multi-algebra subset $\mathcal{S}$ to another subset $\mathcal{S}'$. If it holds that (C1) $\mathcal{S}$ is algebraically stable by H and (C2) algebraically consistent networks over $\mathcal{S}'$ are consistent, then algebraically consistent networks over $\mathcal{S}$ are consistent.*

If, in addition, $\mathcal{S}$ is a $\Rsh$-closed subclass then $\mathcal{S}$ is algebraically tractable.

5 Illustrative Applications of the Theorems

We apply our framework to recover the tractability results of QST (Ex. 1, 12). This is only meant as a simple illustration of our results; obviously, the main interest of our work is that it also applies to networks over large multi-algebras, such as temporal sequences, and not only to bi-networks, but such an application would be more complex and thus less useful as an example. We begin with the combination of $\mathrm{PA}_{\max} = \{<, =, >, \neq, \mathcal{B}\}$ (the maximal distributive subalgebra of PA containing "$\neq$") and $\mathrm{RCC8}_{\max}$, the (non-convex) maximal distributive subalgebra of RCC8 (Long and Li 2015):

Corollary 27. $\mathrm{RCC8}_{\max} \times \mathrm{PA}_{\max}$ *is algebraically tractable.*

Proof. To prove this result, we weaken the projection from PA to RCC8:[2] we consider that $\Rsh_{\mathrm{PA}}^{\mathrm{RCC8}} b = \mathcal{B}_{\mathrm{RCC8}}$ for all

[2]Note that the originally published proof was flawed, because the subclass is actually not simple. This fixed version uses weakened projections, thanks to which simplicity is verified.

$b \in \mathcal{B}_{\mathrm{PA}}$. We prove in the following that the subclass is algebraically tractable for the weakened multi-algebra, which directly entails that it is also algebraically tractable for the original multi-algebra.

We apply the slicing theorem to the subclass with weakened projections, using identity functions as refinements ($\forall r \in \mathcal{S}_i\colon h_i(r) = r$). Scenarizability by h_i (C1) holds since $\diamond$-consistent networks are minimal for $\mathrm{RCC8}_{\max}$ and for $\mathrm{PA}_{\max}$ (Long and Li 2015); simplicity (C2) can be checked by enumeration; $\Rsh$-consistent relations R are consistent (C3) (because we have $\forall b \in R_{\mathrm{PA}}\colon \exists b' \in R_{\mathrm{RCC8}}\colon b \in \Rsh^{\mathrm{PA}}_{\mathrm{RCC8}} b'$ since $R_{\mathrm{PA}} \subseteq \Rsh^{\mathrm{PA}}_{\mathrm{RCC8}} R_{\mathrm{RCC8}}$, and because (b', b) is consistent); finally, algebraically closed scenarios are consistent. Indeed, we can easily prove that the algebraically closed scenarios for the weakened multi-algebra are algebraically closed for the original multi-algebra (any basic relation of $\mathrm{RCC8} \times \mathrm{PA}$ closed under these weakened projections is closed under the original projections). This is sufficient because the algebraically closed scenarios for the original multi-algebra are consistent (Gerevini and Renz 2002). We get the result by $\Rsh$-closure of the subclass. $\square$

Let us now consider H_8, C_8 and Q_8, the three maximal tractable subalgebras of RCC8. This time, we cannot apply the slicing theorem, because closing under projection then under composition does not compute the algebraic closure. However, we can apply the refinement theorem:

Corollary 28. *Let $\mathcal{S}$ be* H_8, C_8, *or* Q_8. *Each subclass $\mathcal{S} \times \mathrm{PA}$ is algebraically tractable.*

Proof. We apply Theorem 26; we use for each $\mathcal{S}$ the classical refinement to basic relations (Gerevini and Renz 2002), denoted $h_{\mathcal{S}}$, and the refinement $h_{\max}$ for PA; $H = (h_{\mathcal{S}}, h_{\max})$ sends relations to $\mathrm{RCC8}_{\max} \times \mathrm{PA}_{\max}$. Each subclass $\mathcal{S} \times \mathrm{PA}$ is algebraically stable by H (C1), as both closure under projection and $\diamond$-consistency are preserved (checked by enumeration; see also Renz (1999)). Algebraically consistent networks over $\mathrm{RCC8}_{\max} \times \mathrm{PA}_{\max}$ are consistent (C2) by Cor. 27. This concludes the proof ($\mathcal{S} \times \mathrm{PA}$ is $\Rsh$-closed). $\square$

Reasoning about descriptions over these subclasses can thus be done efficiently thanks to the algebraic closure. In fact, similarly to the classical case, the algebraic closure can improve reasoning even for intractable subclasses, by improving pruning in backtracking search procedures.

6 Conclusion

We propose a general framework for qualitative constraint networks over combinations of formalisms. It provides a unified way for studying loose integrations and a family of spatio-temporal formalisms, and is also well suited to knowledge representation and reasoning in the context of these combinations. The most notable results of this paper are two complementary theorems entailing the tractability of consistency checking, which we applied to recover the tractability results of the qualitative size and topology combination. Future work will show how our framework also applies to multi-scale descriptions and will introduce several results that we obtained thanks to our theorems, such as the tractability of the preconvex subclass for multi-scale reasoning over the interval algebra.

References

Allen, J. F. 1983. Maintaining knowledge about temporal intervals. *Commun. ACM* 26(11):832–843.

Bennett, B.; Cohn, A. G.; Wolter, F.; and Zakharyaschev, M. 2002. Multi-dimensional modal logic as a framework for spatio-temporal reasoning. *Applied Intelligence* 17(3):239–251.

Chen, J.; Cohn, A. G.; Liu, D.; Wang, S.; Ouyang, J.; and Yu, Q. 2015. A survey of qualitative spatial representations. *The Knowledge Engineering Review* 30(01):106–136.

Chittaro, L., and Montanari, A. 2000. Temporal representation and reasoning in artificial intelligence: Issues and approaches. *Annals of Mathematics and Artificial Intelligence* 28(1-4):47–106.

Cohen-Solal, Q.; Bouzid, M.; and Niveau, A. 2015. An algebra of granular temporal relations for qualitative reasoning. In *Proc. of IJCAI*, 2869–2875.

Cohn, A. G.; Li, S.; Liu, W.; and Renz, J. 2014. Reasoning about topological and cardinal direction relations between 2-dimensional spatial objects. *Journal of Artificial Intelligence Research* 51:493–532.

Freksa, C. 1991. Conceptual neighborhood and its role in temporal and spatial reasoning. In *Proc. of the IMACS Workshop on Decision Support Systems and Qualitative Reasoning*, 181–187.

Gerevini, A., and Nebel, B. 2002. Qualitative spatio-temporal reasoning with RCC-8 and Allen's interval calculus: Computational complexity. In *Proc. of ECAI*, volume 2, 312–316.

Gerevini, A., and Renz, J. 2002. Combining topological and size information for spatial reasoning. *Artificial Intelligence* 137(1):1–42.

Hobbs, J. R. 1985. Granularity. In *Proc. of IJCAI*, 432–435.

Jonsson, P., and Krokhin, A. 2004. Complexity classification in qualitative temporal constraint reasoning. *Artificial Intelligence* 160(1):35–51.

Ladkin, P. B., and Reinefeld, A. 1992. Effective solution of qualitative interval constraint problems. *Artificial Intelligence* 57(1):105–124.

Li, S., and Cohn, A. G. 2012. Reasoning with topological and directional spatial information. *Computational Intelligence* 28(4):579–616.

Li, S., and Nebel, B. 2007. Qualitative spatial representation and reasoning: A hierarchical approach. *The Computer Journal* 50(4):391–402.

Ligozat, G. 1996. A new proof of tractability for ORD-Horn relations. In *Proc. of AAAI/IAAI*, volume 1, 395–401.

Ligozat, G. 2013. *Qualitative spatial and temporal reasoning*. John Wiley & Sons.

Long, Z., and Li, S. 2015. On distributive subalgebras of qualitative spatial and temporal calculi. In *Spatial Information Theory*. Springer. 354–374.

Meiri, I. 1996. Combining qualitative and quantitative constraints in temporal reasoning. *Artificial Intelligence* 87(1–2):343–385.

Nebel, B., and Bürckert, H.-J. 1995. Reasoning about temporal relations: a maximal tractable subclass of Allen's interval algebra. *Journal of the ACM* 42(1):43–66.

Randell, D. A.; Cui, Z.; and Cohn, A. G. 1992. A spatial logic based on regions and connection. In *Proc. of KR*, 165–176.

Renz, J., and Ligozat, G. 2005. Weak composition for qualitative spatial and temporal reasoning. In *Proc. of CP*. Springer. 534–548.

Renz, J. 1999. Maximal tractable fragments of the region connection calculus: A complete analysis. In *Proc. of IJCAI*, 448–455.

van Beek, P., and Cohen, R. 1989. *Approximation algorithms for temporal reasoning*. University of Waterloo. Department of Computer Science.

Vilain, M.; Kautz, H.; and van Beek, P. 1989. Constraint propagation algorithms for temporal reasoning: A revised report. In Weld, D. S., and de Kleer, J., eds., *Readings in Qualitative Reasoning About Physical Systems*. Morgan Kaufmann Publishers Inc. 373–381.

Westphal, M., and Woelfl, S. 2008. Bipath consistency revisited. In *Proc. of the ECAI Workshop on Spatial and Temporal Reasoning*.

Westphal, M.; Hué, J.; Wölfl, S.; and Nebel, B. 2013. Transition constraints: A study on the computational complexity of qualitative change. In *Proc. of IJCAI*, 1169–1175.

Wölfl, S., and Westphal, M. 2009. On combinations of binary qualitative constraint calculi. In *Proc. of IJCAI*, 967–973.