Checking the Consistency of Combined Qualitative Constraint Networks

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Abstract
We study the problem of consistency checking for constraint networks over combined qualitative formalisms. We propose a framework which encompasses loose integrations and a form of spatio-temporal reasoning. In particular, we identify sufficient conditions ensuring the polynomiality of consistency checking, and we use them to find tractable subclasses.

1 Introduction
Temporal and spatial reasoning is omnipresent in our daily lives. Computers can achieve them using quantitative approaches; however, for human-computer interaction, quantitative data is often unavailable or unnecessary. This is why research has been carried out about qualitative approaches to temporal and spatial reasoning – such as the interval algebra of Allen (1983) – not only in artificial intelligence but also in geographical information systems, databases, and multimedia (Chittaro and Montanari 2000; Chen et al. 2015). Some recent research has focused on the combination of qualitative approaches in order to increase their number of applications. One of the most popular combinations is loose integration (Wöllfl and Westphal 2009). Spatio-temporal formalisms are other kinds of combinations (Ligozat 2013), allowing the processing of temporal sequences of spatial information (Westphal et al. 2013). Multi-scale reasoning, the ability to reason at different levels of detail, is also a form of combination (Hobbs 1985; Li and Nebel 2007; Cohen-Solal, Bouzid, and Niveau 2015).

This paper introduces a formal framework capturing the common structure of loose integrations, multi-scale representations, and temporal sequences of spatial information; all of these can indeed be seen as tuples of constraint networks having interdependencies. With loose integrations, each network is based on a different formalism, whereas with multi-scale and spatio-temporal representations all networks are based on the same formalism but hold on different scales and different time periods, respectively. Moreover, in each case, constraints in one network of the tuple can entail constraints between the same variables in the other networks. The entailed constraints correspond to how the initial constraints are transformed by formalism change, scale change, or temporal transition, respectively.

We study in particular consistency checking in the context of our framework; we focus on general results that are common to the three kinds of combination, using a simple, well-known instance of a loose integration as a running example. Specifically, we identify sufficient conditions so that the generalized algebraic closure can be used to check the consistency of networks over some subclass – which is therefore tractable. To sum up, we propose a framework for representing knowledge, reasoning, and identifying tractable fragments, in a unified way, for the three kinds of combination; however, for space reasons, this paper only applies it to loose integrations and spatio-temporal sequences.

We begin by recalling concepts related to temporal and spatial formalisms, then we give some background on combinations of formalisms, notably loose integrations and spatio-temporal sequences. Section 3 introduces our framework and Section 4 establishes our tractability results, which are then illustrated on the combination of size and topology.

2 Background and Related Work
Qualitative Temporal and Spatial Formalisms
In the context of qualitative temporal and spatial reasoning, we are particularly interested in checking the consistency of temporal or spatial descriptions, encoded by relations between spatial or temporal entities of a set \( U \). Each relation is a set of basic relations from a set \( \mathcal{S} \): this represents the uncertainty about the actual basic relation – e.g., \( x \{<,=\} y \) means that either \( x < y \) or \( x = y \). The set of all relations forms a non-associative relation algebra \( \mathcal{A} = 2^\mathcal{S} \) (Ligozat 2013, Ch. 11). Well-known algebras include the interval algebra of Allen (1983), but also the point algebra PA (Vilain, Kautz, and van Beek 1989), whose basic relations are \( \mathcal{B}_{PA} = \{<,=,>\} \), and the algebra RCC8 of topological relations (Randell, Cui, and Cohn 1992), whose basic relations are described in Fig. 1. There are several operators over relations in \( \mathcal{A} \): inverse of a relation \( r \), denoted by \( \text{INV} \); intersection of \( r_1 \) and \( r_2 \), denoted by \( r_1 \cap r_2 \); and (weak) composition of \( r_1 \) and \( r_2 \) (Renz and Ligozat 2005), denoted by \( r_1 \odot r_2 \). These operators allow one to reason, by reducing the uncertainty about basic relations of entities: \( x \, r \, y \iff y \, \text{INV} \, x \);
In the literature, consistency (in polynomial time) by repeatedly replacing each network solution by computing its algebraic closure, resulting network is not trivially inconsistent (i.e., if none of the networks having interdependencies) (Westphal and Woelfl 2008). The consistency checking problem is then to decide whether there is a solution satisfying both networks.

Example 1. The loose integration of qualitative size and topology of Gerevini and Renz (2002), which we call QST, describes the relation between two regions both in terms of topology and in terms of their relative size; e.g., “x and y are disjoint and the size of x is smaller than that of y”.

To reason on QST, Gerevini and Renz generalized the path-consistency algorithm, which simply computes the algebraic closure, into the bipath-consistency algorithm, which enforces o-consistency on both networks while simultaneously propagating their interdependencies. Subclasses for which bipath-consistency decides consistency have been found for several combinations of formalisms (Gerevini and Renz 2002; Li and Cohn 2012; Cohn et al. 2014).

The framework introduced in this paper encompasses loose integrations (generalized to m formalisms) as specific combinations, but does not cover all ways of combining formalisms. Tight integrations (Wölfl and Westphal 2009) are more expressive than loose integrations, at the cost of drastically increasing the number of relations. Another combination is that of Meiri (1996), which deals with heterogeneous entities that are points and intervals. The corresponding relations are relations between a point and an interval, two intervals, and two points, respectively. In this combination, whose complexity has been studied in depth by Jonsen and Krokhin (2004), there is only one relation per pair of entities; in contrast, loose integrations feature several relations (from different formalisms) between the same entities, which increases expressiveness by allowing the use of complementary relations. This complementarity is the main asset of loose integration (and its major difficulty). Note that there also exist combinations with non-qualitative formalisms (Meiri 1996; Bennett et al. 2002)

**Spatio-Temporal Formalisms**

Spatio-temporal formalisms are also combinations, which integrate space and time information in particular ways. Westphal et al. (2013) proposed a method to reason about temporal sequences of spatial information, which actually share the same structure as loose integrations and are thus covered by our framework. They model such sequences as

![Figure 1: The 8 relations of RCC8 in the plane.](image)
tuples of constraint networks, each corresponding to a time instant. They introduce two kinds of solutions, depending on the desired dynamics of entities (moving continuously) over time. Our framework covers the weaker “T_2-solutions”, which guarantee that between successive instants of the sequence, for each pair of entities, only the relation of the first instant and then the relation of the second instant hold.

**Example 2.** The following is a temporal sequence of 3 networks describing spatial points moving along a line: $x \leq y < z$ at the first instant, $x = y = z$ at the second, and then $x > y > z$ at the third instant. This description has temporally continuous solutions without intermediary relations between the instants, such as that of Figure 2 (a).

In fact, the $T_2$ condition forces relations at successive instants to be “neighbors” according to the neighborhood graph of PA (Frenska 1991), shown in Figure 2 (b). In this graph, for example, the only neighbor relation of “$<$” is “$=$”. The $T_2$ condition thus ensures that, if $x < y$ at one instant, then at any neighbor instant, either $x < y$ or $x = y$.

Gerevini and Nebel (2002) proposed a similar formalism based on time intervals but with uncertainty on the scheduling of intervals, so it is not encompassed by our framework.

### 3 Representation and Reasoning with Multi-Algebras

This section introduces multi-algebras, which constitute the underlying structure of loose integrations and temporal sequences. It shows how one can reason about multi-algebra relations, then provides them with a formal semantics, and finally generalizes constraint networks and their algebraic closure to this broader setting.

#### Projections and Multi-Algebras

Let us introduce the general building blocks of our framework, beginning with projections, which aim at representing the interdependencies of relations from different formalisms. The projection of a relation $r$ from a formalism onto another is the set of basic relations of the other formalism which may hold given that $r$ holds.

**Definition 3.** Let $A = 2^B$ and $A' = 2^{B'}$ be two algebras. A projection operator is a function $\tau : A \rightarrow A'$ which satisfies (i) $\forall b \in B$, $\tau \{ b \} = \tau' \{ b \}$, and (ii) $\forall r \in A$, $\tau r = \bigcup_{b \in B} \tau r \{ b \}$.

We can now define multi-algebras, the key objects of our framework, which are Cartesian products of algebras (each corresponding to one of the combined formalisms, or one instant in a temporal sequence) associated with projection operators representing the interdependencies of their relations.

**Definition 4.** A multi-algebra $A$ is the Cartesian product of $m$ algebras $A_1, \ldots, A_m$ (with $m \in \mathbb{N}^*$), equipped with $m(m-1)$ projection operators $\tau_{ij} : A_i \rightarrow A_j$ (for any distinct $i, j \in \{1, \ldots, m\}$). We call relations the elements $R$ of $A$, although they are actually $m$-tuples of relations; $R_i$ denotes the (classical) relation associated with $A_i$ in $R$. We say that $R$ is basic when all $R_i$ are basic ($R \in B_1 \times \cdots \times B_m$).

Note that a multi-algebra with $m = 1$ (“mono-algebra”) is exactly a classical algebra, as it has no projection operators.

**Example 5.** The multi-algebra corresponding to QST (see Ex. 1) is the Cartesian product $\text{RCC}8 \times \text{PA}$ of the RCC8 algebra (see Fig. 1) and the point algebra PA (for region sizes), with the interdependency operators of QST as projections. One of its relations is $(\{\text{TPP}\}, \{\leq, =\})$, and the projection of $(\text{TPP})$ into PA is $\tau_{\text{PA}}\text{RCC}8\{\text{TPP}\} = \{=\}$ (since TPP is the “tangential proper part” relation and a region strictly included in another always has a smaller size).

**Example 6.** The multi-algebra $P_{\text{PA}}m$ can be used to represent sequences of binary relations between points on a line, as in Ex. 2: the $i^{th}$ PA corresponds to the $i^{th}$ instant of the sequence, thus $R_i$ is the relation at instant $i$. The projections enforcing the neighborhood graph of Fig. 2 are $\tau_i^1\{<\} = \{<, =\}$, $\tau_i^2\{>\} = \{>, =\}$, and $\tau_i^3\{=\} = B$, if instants $i$ and $j$ are neighbors (i.e., $|i - j| = 1$), and $\forall b \in B : \tau_i^r\{b\} = B$ (i.e., no constraint), if they are not. For instance, $\text{PA} \times \text{PA} \times \text{PA}$ is the multi-algebra corresponding to three instants. The relation $(\{<, =\}, \{=\}, \{>, =\})$ of this multi-algebra represents a possible 3-instant sequence of relations.

When there is no ambiguity, we use a lighter notation for relations, thus writing $(\{\text{TPP}, \text{EQ}\}, \{<\})$ as $\text{TPP} \cdot \text{EQ} <$, and $(\{<, =\}, \{=\}, \{>, =\})$ as $(\leq, =, \neq)$, for example.

### Reasoning about Multi-Algebra Relations

We can reason about multi-algebra relations by applying the classical rules componentwise: for instance, in QST (Ex. 5), if $x (\text{TPP}, \leq) y$ and $y (\text{DC}, =) z$, then $x (\text{DC}, \leq) z$ (since $\text{TPP} \circ \text{DC}$ is $\text{DC}$ and $\leq \circ = = \leq$). It is thus natural to introduce composition $\circ$, intersection $\cap$, and inversion $\cdot$ operators over multi-algebra relations which simply work componentwise (e.g., $(R \circ R')_i = R_i \circ R'$. These operators are also useful to apply classical concepts to our generalized framework (for the same reason, we also write $R \subseteq R'$ if $R_i \subseteq R'_i$ for each $i$). They are, however, not sufficient for reasoning: we also need to propagate the interdependencies inside each relation.

**Definition 7.** The projection closure of $R \in A$, denoted by $\tau_i(R)$, is obtained from $R$ by repeatedly replacing each $R_j$ by $R_i \cap (\tau_i R_j)$ for all distinct $i, j$, until a fixed point is reached.

**Example 8.** In QST, since $\tau_{\text{PA}}\text{RCC}8\{\text{TPP}\} = \{<\}$ (Ex. 5), the projection closure of $(\text{TPP}, \leq)$ is $\tau_i (\text{TPP}, \leq) = (\text{TPP}, <)$, and also $\tau_i (\text{TPP}, \geq) = (\emptyset, \emptyset, \emptyset)$, which proves that this relation is not feasible (indeed, a region cannot be inside another while having a larger surface). In the context of Ex. 6, $\tau_i (\leq, \neq, >) = (\emptyset, \emptyset, \emptyset)$, so this 3-instant sequence is not feasible.
Semantics and Consistency of Relations

Using these operators, we can now give multi-algebra relations a proper semantics, taking an approach similar to the classical case:

Definition 9. A (loosely) combined qualitative formalism is a triple \((\mathcal{A}, U, \varphi)\), where \(\mathcal{A}\) is a multi-algebra, \(U\) is an entity domain, and the interpretation \(\varphi: \mathcal{A} \to 2^U \times U\) satisfies:

\[
\varphi(\mathcal{R}) = \varphi(R) \quad \varphi(R \circ R') \supseteq (\varphi(R) \circ \varphi(R')) \cap \varphi(B)
\]

\[
\varphi(\mathcal{R}) = \varphi(R) \quad \varphi(R \cap R') = \varphi(R) \cap \varphi(R')
\]

\[
\varphi(\mathcal{R}) = \varphi(R) \quad \varphi(R \cup R') \supseteq \varphi(R) \cup \varphi(R')
\]

with \(R, R' \in \mathcal{A}\) and \(\circ\) the true composition of relations on \(U\).

These straightforward requirements ensure that the operators are sound (i.e., do not remove valid pairs of entities), and that a relation is consistent (i.e., has a nonempty interpretation) if and only if it contains a consistent basic relation.

Example 10. We call temporaliﬁed point calculus (TPC) the combined formalism representing temporal sequences of the point algebra interpreted spatially. Its multi-algebra is simply \(\text{PA}^m\) (for sequences of length \(m\), as described in Ex. 6. Its interpretation function (which we cannot deﬁne formally for space reasons) associates with each relation the set of pairs of points evolving continuously on \(\mathbb{R}\) along the time of the sequence, satisfying at each instant of the sequence the corresponding relation, and not satisfying other relations between these instants (this is the \(T_2\) condition; see Ex. 2).

The combined formalism corresponding to the loose integration of \(m\) formalisms is straightforward to deﬁne (it can be checked that all requirements of Def. 9 are satisﬁed):

Definition 11. The loose integration of \(m\) qualitative formalisms \((\mathcal{A}_1, U, \varphi_1), \ldots, (\mathcal{A}_m, U, \varphi_m)\) over the same domain \(U\) is the combined formalism \((\mathcal{A}, U, \varphi)\), where \(\mathcal{A}\) is the multi-algebra \(\mathcal{A}_1 \times \cdots \times \mathcal{A}_m\) with each \(r_i\) satisfying \(r_i^b = \{b' \in B_i \mid \varphi_i(b') \cap \varphi_i(b) \neq \emptyset\}\), and \(\varphi_r: (r_1, \ldots, r_m) \to \varphi_i(r_1) \cap \cdots \cap \varphi_m(r_m)\).

Example 12. QST is exactly the loose integration of the RCC8 formalism and of the formalism interpreting PA in terms of region sizes; we recover the multi-algebra of Ex. 5. On the other hand, it can be shown that TPC (Ex. 10) is not a loose integration (the interpretations at every instant cannot be deﬁned independently from one another).

Now, while in classical formalisms any basic relation is consistent, this is no longer the case in combined formalisms: because of interdependencies, multi-algebra relations (even basic ones) can be inconsistent. In particular, we have seen (Ex. 8) that the projection closure of a relation can be empty. Closing a relation under its projection operators can thus help detecting its inconsistency: if \(\mathcal{R}\) is empty, we can conclude that \(\mathcal{R}\) is inconsistent. Otherwise, \(\mathcal{R}\) is “consistent with respect to projections”, or \(\mathcal{R}\)-consistent.

Definition 13. A multi-algebra relation \(R\) is \(\mathcal{R}\)-consistent if \(\mathcal{R} = R^\mathcal{R}(R)\) and \(R_i \neq \emptyset\) for each \(i\).

However, observe that while projections can remove pairwise inconsistencies, consistency ultimately depends on the interpretation function. Hence, \(\mathcal{R}\)-consistency does not imply consistency in general, although it is the case for some combined formalisms: we can prove in particular that any \(\mathcal{R}\)-consistent relation of TPC and QST (Ex. 10, 12) is consistent. In such cases, closing a relation by projection suﬃces to check its consistency.

Multi-Algebra Networks and Algebraic Closure

We simply model descriptions from combined formalisms as qualitative constraint networks over multi-algebras, which work exactly like classical networks except that constraints between entity variables are \(m\)-tuples of relations. For example, the network \(N\) in Fig. 3 corresponds to the temporal sequence of 3 networks over the point algebra shown in Ex. 2; relation \(N^\mathcal{R}\) is the sequence of relations between \(x\) and \(y\). It is important to note that \(N\) can equivalently be seen as a tuple of classical networks over each \(\mathcal{A}_i\), as shown in Fig. 3, where \(N_i\) is the network at instant \(i\).

Definition 14. Let \(N = (E, C)\) be a network over some multi-algebra \(\mathcal{A}\). The \(i\)th slice of \(N\), denoted by \(N_i\), is the network \((E, C_i)\) over \(\mathcal{A}_i\), where \(C_i = \{(x, R_i, y) \mid (x, y) \in C\}\).

We directly adapt the notions of solution, consistency and scenario (Sect. 2) to networks over multi-algebras (with respect to a combined formalism; we often omit this precision when doing so is harmless). For instance, a solution of the network in Fig. 3 is the evolution in Fig. 2 (a). Other notions must be generalized:

Definition 15. A network \(N\) over a multi-algebra \(\mathcal{A}_1 \times \cdots \times \mathcal{A}_m\) is trivially inconsistent if \(\exists i \in \{1, \ldots, m\} : \exists x, y \in E : N_i^{\mathcal{R}} = \emptyset\). It is algebraically closed if \(N_i^{\mathcal{R}} \subseteq N_i^{\mathcal{R}} \circ N_i^{\mathcal{R}}\) and \(N_i^{\mathcal{R}} = \mathcal{R} N_i^{\mathcal{R}}\) for all \(x, y, z \in E\). It is algebraically consistent if it is algebraically closed and not trivially inconsistent.

For networks over multi-algebras, being algebraically closed is being closed under both composition and projection. This cleanly generalizes the classical case, since any
relation of a mono-algebra is vacuously closed under projection.

Algebraic consistency generalizes bipath-consistency to \( m \) dimensions, and works in the same way – it is a necessary condition for consistency that can be used to filter out inconsistent networks. It can be enforced by alternately closing each relation \( N_{\text{TPP}} \) under projection and each slice \( N_i \) under composition until a fixed point is reached.

By using the construction in \( \text{Def. 19} \), it is actually inconsistent: First, \( \text{\textnormal{\texttt{NTPP}}} \) of \( \text{\textnormal{\texttt{N}}} \) not belong to any consistent scenario of \( \mathcal{A} \). Each relation \( \text{\textnormal{\texttt{N}}} \) of a mono-algebra is vacuously closed under projection, as \( \{ \text{\textnormal{\texttt{NTPP}}} \notin N_{\text{PA}} \} \notin N_{\text{RCCS}} \). Now, while \( N \) is algebraically consistent (and although \( N_{\text{PA}} \) and \( N_{\text{RCCS}} \) are consistent), it is actually inconsistent: First, \( \text{\textnormal{\texttt{N}}} \) of \( N_{\text{PA}} \) does not belong to any consistent scenario of \( N_{\text{PA}} \) (van Beek and Cohen 1989, p. 13). Second, \( \text{\textnormal{\texttt{NTPP}}} \) of \( \text{\textnormal{\texttt{N}}} \) is not feasible either (for \( N_{\text{RCCS}} \)). The only remaining relation, \( \text{\textnormal{\texttt{EQ}}} \) \( \notin \text{\textnormal{\texttt{R}}} \), is also not feasible, since its projection closure is empty.

However, by adding \( \text{\textnormal{\texttt{DC}}} \) to \( N_{\text{RCCS}} \), the network remains algebraically consistent, but this time it becomes consistent. Indeed, Fig. 4 (b) shows one of its consistent scenarios.

### 4 Tractability Results

Let us now study the problem of checking the consistency of networks over multi-algebras. Since this problem is NP-complete for many formalisms, we proceed as in the classical case: we focus on subclasses, and notably on subclasses, of multi-algebras. Then, we present two theorems providing conditions under which a subclass is tractable.

#### Algebraically Tractable Subclasses

We first introduce two kinds of multi-algebra subsets, namely subclasses and the more specific subclasses:

**Definition 17.** A subclass of a multi-algebra \( \mathcal{A} \) is a set of relations \( S \subseteq \mathcal{A} \) which is closed under componentwise composition, intersection, and inversion. If \( S \) contains the basic relations (i.e., \( B_1 \times \cdots \times B_m \subseteq S \)), we call it a subclass.

For example, \( H_{\text{8}} \times \text{PA} \) – where \( H_{\text{8}} \) is a well-known subalgebra of \( \text{RCC8} \) (Gerevini and Renz 2002) – is a subclass of the multi-algebra \( \text{RCC8} \times \text{PA} \). Subalgebras are particularly interesting subclasses since all scenarios of the multi-algebra are scenarios of these subclasses. Moreover, the most studied subclasses are subalgebras (Nebel and Bürckert 1995; Ligotzat 1996; Renz 1999; Long and Li 2015).

The following notion of slice of a multi-algebra subset is a kind of reverse operator of the Cartesian product. It will allow us to lift tractability results from the classical setting to the multi-algebra setting.

**Definition 18.** The \( i \)-th slice of a multi-algebra subset \( S \subseteq \mathcal{A} \), denoted \( S_i \), is the subset of \( \mathcal{A}_i \) defined by \( S_i = \{ R_i \mid R \in S \} \).

Note that \( S \) is a subset of \( \mathcal{A}_1 \times \cdots \times \mathcal{A}_m \), the Cartesian product of its slices. It is also not hard to see that, when \( S \) is a subclass, each slice \( S_i \) is a subclass of \( \mathcal{A}_i \), and the Cartesian product of the \( S_i \) is also a subclass of \( \mathcal{A} \).

Recall that the algebraic closure is classically used to detect inconsistent networks, providing a consistency checking procedure that is polynomial and sound (since the operators are sound, thanks to Def. 9), but incomplete. We focus on subclasses for which the procedure is complete:

**Definition 19.** A subclass \( S \) is said to be algebraically tractable when, for any network \( N \) over \( S \), if the algebraic closure of \( N \) is not trivially inconsistent then \( N \) is consistent.

Clearly, for a subclass to be algebraically tractable, all the algebraically closed scenarios over this subclass must be consistent; this depends on the interpretation function of the combined formalism. For instance, some algebraically closed scenarios over the combination of \( \text{RCC8} \) (with weak connectedness) and the rectangle algebra are inconsistent (Cohn et al. 2014); consequently, no subalgebra of this combination can be algebraically tractable.

One could think that if all algebraically consistent networks over a subclass \( S \) are consistent, then \( S \) is algebraically tractable; but this is not sufficient because, contrary to the classical case, the algebraic closure of a network over a subclass \( S \) is not necessarily over \( S \). It clearly becomes sufficient if \( S \) is \( \rightarrow \)-closed, i.e., if the projection closure of any relation of \( S \) is in \( S \) (\( \forall \mathcal{R} \in \mathcal{S} : \rightarrow \mathcal{R} \in \mathcal{S} \)).

**Proposition 20.** A \( \rightarrow \)-closed subclass over which algebraically consistent networks are consistent is algebraically tractable.

Now, under which conditions does a \( \rightarrow \)-closed subclass verify that its algebraically consistent networks are consistent? In the following we state two complementary theorems providing such conditions. With the first theorem, a subalgebra inherits its tractability from that of its slices, whereas with the second theorem, tractability is inherited from a smaller subset of relations by refinement.

#### Inheriting Tractability from Subalgebra Slices

In this section, we focus on conditions ensuring that a subalgebra \( S \) is tractable by using the tractability of its slices \( S_i \). One of the conditions is that each slice be scenarizable by a refinement (a refinement of a multi-algebra subset \( S \) is a mapping \( h : S \rightarrow \mathcal{A} \) such that \( h(\mathcal{R}) \subseteq \mathcal{R} \) for all \( \mathcal{R} \in \mathcal{S} \)).

**Definition 21.** A mono-subalgebra \( S \) is scenarizable by a refinement \( h \) if for any \( \rightarrow \)-consistent network \( N \) over \( S \) and any \( x,y \in \mathcal{E} \), (i) \( h(\mathcal{N}^x) \neq \emptyset \) and (ii) for any \( b \in h(\mathcal{N}^x) \) there exists an algebraically closed scenario \( S \subseteq N \) such that \( S^y = b \).

When all the algebraically closed scenarios over \( S \) are consistent (see remark after Def. 19), this property actually entails the algebraic tractability of \( S \). Indeed, from any \( \rightarrow \)-consistent network over \( S \), we can obtain a consistent scenario by (i) choosing a pair of variables, (ii) replacing their relation by a basic relation of the refinement, (iii) computing the algebraic closure – and repeating these steps until a scenario is obtained. Finding a refinement by which a subalgebra \( S \) is scenarizable is a classical method to prove that \( S \) is tractable. For example, the point algebra \( \text{PA} \) and the pre-convex subclass of the interval algebra are scenarizable by \( h_{\text{max}} \), the refinement by the basic relations of “maximal dimension” (Ligozat 2013).
Now, considering a subalgebra $S$ whose each slice $S_i$ is scenarizable by some $h_i$, a natural idea would be to apply the classical technique by combining $h_1, \ldots, h_m$ into a specific form of refinement over multi-algebra relations:

**Definition 22.** A multi-refinement of a multi-algebra subset $S$ is a refinement of the form $H = (h_1, \ldots, h_m)$ with each $h_i$ a refinement of $S_i$, defined as $H : R \mapsto (h_1(R_1) \ldots, h_m(R_m))$.

However, even if each $S_i$ is scenarizable by $h_i$, there is in fact no guarantee that the multi-refinement $H = (h_1, \ldots, h_m)$ can be used to find a consistent scenario using an adaptation of the “scenarizability by $h$” method. Additional requirements are needed to ensure that the individual refinements work well together with respect to projections. First, the refinement of $\tau$-consistent relations (Def. 13) by $H$ must be consistent; but this is not sufficient for any algebraically consistent network to remain consistent after refinement. Consequently, we assume in addition the following property:

**Definition 23.** A network is simple when closing it under projection and then under composition makes it either algebraically consistent or trivially inconsistent.

A subalgebra $S$ is simple if any network over $S$ is simple.

Since being closed is a local property, it can be shown that enumerating all 3-variable bi-networks over each $S_i \times S_j$ suffices to check that a subalgebra $S$ is simple.

Using the previous properties, we state our first theorem:

**Theorem 24** (Slicing theorem). Let $S$ be a subalgebra whose algebraically closed scenarios are consistent, and $H = (h_1, \ldots, h_m)$ be a multi-refinement. If we have:

1. (C1) each slice $S_i$ is scenarizable by $h_i$;
2. (C2) $S$ is simple; and
3. (C3) for any $\tau$-consistent $R$ of $S$, $H(R)$ is consistent;

then algebraically consistent networks over $S$ are consistent. If, in addition, $S$ is $\tau$-closed, then $S$ is algebraically tractable.

**Proof.** Let $N$ be an algebraically consistent network over $S$, and $x, y$ be variables such that $N^{xy}$ is not basic. We know that $H(N^{xy})$ is consistent (C3), that is, it contains at least one consistent basic relation (Def. 9). We refine $N^{xy}$ by one such relation $B = (b_1, \ldots, b_m)$. Obviously, the modified $N$, which we denote by $N'$, is still closed under projection.

Moreover, since each slice $N_i$ is $\tau$-consistent, there exists an algebraically closed scenario $S_i \subseteq N_i$ such that $S_i^{xy} = b_i$. Therefore, $S_i \subseteq N_i^{|}$ holds for all $i$, which ensures that the closure of $N'$ under composition is not trivially inconsistent. The closure of $N'$ is thus algebraically consistent, since $S$ is simple (C2) and $N'$ is closed under projection.

All in all, the result is an algebraically consistent network over $S$ (as $S$ is a subalgebra). We can thus apply the procedure from the start once again, iteratively making all relations basic and consistent. The process necessarily ends with an algebraically closed scenario, which is consistent by hypothesis. Hence $N$ is consistent (Def. 9). The second conclusion is a direct corollary (by Prop. 20).

This theorem can be used to prove that a $\tau$-closed subclass $S$ built from known tractable subalgebras is tractable. It also gives a very efficient way to check the consistency of a network over $S$, only requiring one projection closure followed by one composition closure. Moreover, the proof describes an efficient algorithm exhibiting a consistent scenario.

**Inheriting Tractability from a Subset of Relations**

Now, we focus on an alternative set of conditions ensuring the tractability of a $\tau$-closed subclass $S$. This result is inspired by the classical technique of “reduction by a refinement” (Renz 1999). The idea is to inherit, from a smaller multi-algebra subset $S'$, the fact that algebraically consistent networks are consistent – and to conclude with Prop. 20. Indeed, if we can refine networks over $S$ to networks over $S'$ while preserving algebraic consistency, then all algebraically consistent networks over $S$ are consistent. We call this requirement algebraic stability by a refinement:

**Definition 25.** A multi-algebra subset $S$ is algebraically stable by a refinement $H$ if, for any algebraically consistent network $N$ over $S$, the refined network $H(N)$ (obtained from $N$ by simultaneously replacing each relation $N^{xy}$ by $H(N^{xy})$) is still algebraically consistent.

For example, $H_8 \times \text{PA}$ is algebraically stable by $H = (h_1, h_{\text{max}})$, with $h_{\text{max}}$ the $H_8$ refinement of Renz (1999, Lemma 20); like simplicity, stability is easy to check by enumeration.

Our theorem formalizes the reduction mechanism:

**Theorem 26** (Refinement theorem). Let $H$ be a refinement from a multi-algebra subset $S$ to another subset $S'$. If it holds that (C1) $S$ is algebraically stable by $H$ and (C2) algebraically consistent networks over $S'$ are consistent, then algebraically consistent networks over $S$ are consistent.

If, in addition, $S$ is a $\tau$-closed subclass then $S$ is algebraically tractable.

5 Illustrative Applications of the Theorems

We apply our framework to recover the tractability results of QST (Ex. 1, 12). This is only meant as a simple illustration of our results; obviously, the main interest of our work is that it also applies to networks over large multi-algebras, such as temporal sequences, and not only to bi-networks, but such an application would be more complex and thus less useful as an example. We begin with the combination of $\text{PA}_{\text{max}} = \{<,=,\neq,\bigcup,\cap,\exists\}$ (the maximal distributive subalgebra of $\text{PA}$ containing “$\neq$”) and $\text{RCC8}_{\text{max}}$ the (non-convex) maximal distributive subalgebra of $\text{RCC8}$ (Long and Li 2015):

**Corollary 27.** $\text{RCC8}_{\text{max}} \times \text{PA}_{\text{max}}$ is algebraically tractable.

**Proof.** To prove this result, we weaken the projection from $\text{PA}$ to $\text{RCC8}$: we consider that $\tau_{\text{RCC8}} b = B_{\text{RCC8}}$ for all $b \in B_{\text{PA}}$. We prove in the following that the subclass is algebraically tractable for the weakened multi-algebra, which directly entails that it is also algebraically tractable for the original multi-algebra.

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2Note that the originally published proof was flawed, because the subclass is actually not simple. This fixed version uses weakened projections, thanks to which simplicity is verified.
We apply the slicing theorem to the subclass with weakened projections, using identity functions as refinements ($\forall r \in S_i : h_i(r) = r$). Scenarizability by $h_i$ (C1) holds since $\circ$-consistent networks are minimal for $RCC8_{\text{max}}$ and for $PA_{\text{max}}$ (Long and Li 2015); simplicity (C2) can be checked by enumeration; $\overset{r}{\circ}$-consistent relations $R$ are consistent (C3) (because we have $\forall b \in R_{PA} : \exists b' \in R_{RCC8} : b \in R_{PA}RCC8 b'$ since $R_{PA} \subseteq R_{PA}RCC8$, and because $(b', b)$ is consistent); finally, algebraically closed scenarios are consistent. Indeed, we can easily prove that the algebraically closed scenarios for the weakened multi-algebra are algebraically closed for the original multi-algebra (any basic relation of $RCC8 \times PA$ closed under these weakened projections is closed under the original projections). This is sufficient because the algebraically closed scenarios for the original multi-algebra are consistent (Gerevini and Renz 2002). We get the result by $\overset{r}{\circ}$-closure of the subclass.

Let us now consider $H_8$, $C_8$ and $Q_8$, the three maximal tractable subalgebras of $RCC8$. This time, we cannot apply the slicing theorem, because closing under projection then under composition does not compute the algebraic closure. However, we can apply the refinement theorem:

**Corollary 28.** Let $S$ be $H_8$, $C_8$ or $Q_8$. Each subclass $S \times PA$ is algebraically tractable.

**Proof.** We apply Theorem 26; we use for each $S$ the classical refinement to basic relations (Gerevini and Renz 2002), denoted $h_S$, and the refinement $h_{\text{max}}$ for $PA$; $H = (h_S, h_{\text{max}})$ sends relations to $RCC8_{\text{max}} \times PA_{\text{max}}$. Each subclass $S \times PA$ is algebraically stable by $H$ (C1), as both closure under projection and $\circ$-consistency are preserved (checked by enumeration; see also Renz (1999)). Algebraically consistent networks over $RCC8_{\text{max}} \times PA_{\text{max}}$ are consistent (C2) by Cor. 27. This concludes the proof ($S \times PA$ is $\overset{r}{\circ}$-closed).

Reasoning about descriptions over these subclasses can thus be done efficiently thanks to the algebraic closure. In fact, similarly to the classical case, the algebraic closure can improve reasoning even for intractable subclasses, by improving pruning in backtracking search procedures.

### 6 Conclusion

We propose a general framework for qualitative constraint networks over combinations of formalisms. It provides a unified way for studying loose integrations and a family of spatio-temporal formalisms, and is also well suited to knowledge representation and reasoning in the context of these combinations. The most notable results of this paper are two complementary theorems entailing the tractability of consistency checking, which we applied to recover the tractability results of the qualitative size and topology combination. Future work will show how our framework also applies to multi-scale descriptions and will introduce several results that we obtained thanks to our theorems, such as the tractability of the preconvex subclass for multi-scale reasoning over the interval algebra.

### References


